§5. Topological Phenomena

§5.1 Magnetic Monopoles

U(1) gauge theory

Let M be a Riemannian manifold.
Consider a $U(1)$ bundle $P \xrightarrow{\pi} M$
Then P is trivial if M is contractible to a point

$\longrightarrow$ for $M = \mathbb{R}^4$ we have

$$P = \mathbb{R}^4 \times U(1)$$

The gauge potential (connection) is simply

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu \quad (\text{note: } \mathcal{A}_\mu = i A_\mu)$$

$\rightarrow$ field strengh $\mathcal{F} = d\mathcal{A}$ is

$$\mathcal{F}_{\mu\nu} = \frac{\partial \mathcal{A}_\nu}{\partial x^\mu} - \frac{\partial \mathcal{A}_\mu}{\partial x^\nu}$$

and $d\mathcal{F} = \mathcal{F} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F} = 0$

in components: $\partial_\lambda \mathcal{F}_{\mu\nu} + \partial_\nu \mathcal{F}_{\lambda\mu} + \partial_\mu \mathcal{F}_{\nu\lambda} = 0 \quad (1)$

Identifying the components as

$$\mathcal{F}_{\mu\nu} := i\, F_{\mu\nu}$$

with $E_i = F_{i0}$, $B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}$ $(i,j,k = 1,2,3)$

→ (1) becomes:

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 , \quad \vec{\nabla} \cdot \vec{B} = 0 \qquad (2)$$

The Maxwell action $S[\mathcal{A}]$ is given by

$$S[\mathcal{A}] := \frac{1}{4} \int_{\mathbb{R}^4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} d^4x$$

$$= -\frac{1}{4} \int_{\mathbb{R}^4} F_{\mu\nu} F^{\mu\nu} d^4x$$

Define $* \mathcal{F}_{\mu\nu} = \frac{1}{2} \mathcal{F}^{\kappa\lambda} \varepsilon_{\kappa\lambda\mu\nu}$ as the "dual"

$$\longrightarrow \quad S[\mathcal{A}] = -\frac{1}{4} \int_{\mathbb{R}^4} \mathcal{F} \wedge * \mathcal{F} \qquad \text{(exercise)}$$

Variation with respect to $\mathcal{A}_\mu$ gives

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \qquad (3)$$

giving $\quad \vec{\nabla} \cdot \vec{E} = 0 , \quad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0 \qquad (4)$

Consider now the combination $\vec{E} + i\vec{B}$
then (2) and (4) are invariant
under

$$(\vec{E} + i\vec{B}) \longmapsto e^{i\theta}(\vec{E} + i\vec{B})$$

i.e.
$$\vec{E} \longmapsto \cos\theta\,\vec{E} - \sin\theta\,\vec{B}$$
$$\vec{B} \longmapsto \cos\theta\,\vec{B} + \sin\theta\,\vec{E}$$

Check: $\vec{\nabla}\cdot(\vec{E}_\theta) = \vec{\nabla}\cdot\vec{B}_\theta = 0$ ✓

$$\vec{\nabla}\times \vec{E}_\theta = \cos\theta\,\vec{\nabla}\times\vec{E} - \sin\theta\,\vec{\nabla}\times\vec{B}$$
$$= -\cos\theta\,\frac{\partial\vec{B}}{\partial t} - \sin\theta\,\frac{\partial\vec{E}}{\partial t}$$
$$= -\frac{\partial}{\partial t}\vec{B}_\theta$$

$$\vec{\nabla}\times\vec{B}_\theta = \cos\theta\,\vec{\nabla}\times\vec{B} + \sin\theta\,\vec{\nabla}\times\vec{E}$$
$$= \cos\theta\,\frac{\partial}{\partial t}\vec{E} - \sin\theta\,\frac{\partial\vec{B}}{\partial t}$$
$$= \frac{\partial}{\partial t}\vec{E}_\theta$$

The Dirac magnetic monopole

Let us extend the above to $U(1)$ bundles over non-trivial base

→ Dirac monopole on $\mathbb{R}^3 - \{0\}$ homeomorphic to S^2

→ relevant bundle: $P(S^2, U(1))$

S^2 is covered by two charts:

$$U_N := \left\{ (\theta, \phi) \mid 0 \leq \theta \leq \frac{1}{2}\pi + \varepsilon \right\}$$

$$U_S := \left\{ (\theta, \phi) \mid \frac{1}{2}\pi - \varepsilon \leq \theta \leq \pi \right\}$$

where θ and ϕ are polar coordinates

the corresponding connections are giben by

$$\mathcal{A}_N = ig(1 - \cos\theta)\, d\phi, \qquad \mathcal{A}_S = -ig(1 + \cos\theta)\, d\phi$$

where g is the monopole charge

To see how this comes about, consider a monopole of charge g sitting at $\vec{r} = 0$:

$$\vec{\nabla} \cdot \vec{B} = 4\pi g\, \delta^{(3)}(\vec{r})$$

$$\Delta(1/r) = -4\pi\, \delta^{(3)}(\vec{r}) \quad \text{and} \quad \vec{\nabla}(1/r) = -\frac{\vec{r}}{r^3}$$

$$\longrightarrow \quad \vec{B} = g \frac{\vec{r}}{r^3}$$

magnetic flux Φ is given by

$$\Phi = \oint_S \vec{B} \cdot d\vec{S} = 4\pi g$$

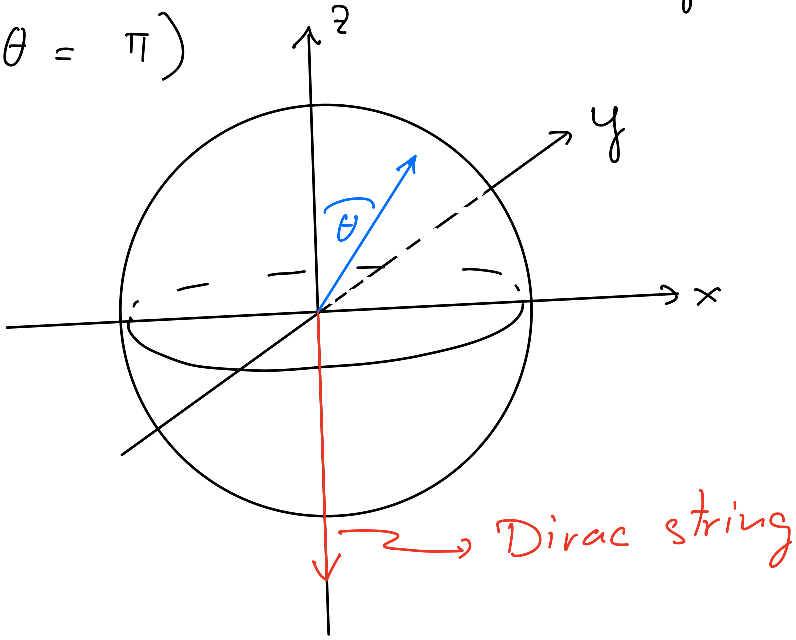
sphere with radius R

Next, define the vector potential by

$$A^N_x = \frac{-gy}{r(r+z)}, \quad A^N_y = \frac{gx}{r(r+z)}, \quad A^N_z = 0$$

$$\longrightarrow \quad \vec{\nabla} \times \vec{A}^N = \frac{g\vec{r}}{r^3} + 4\pi g\, \delta(x)\delta(y)\,\theta(-z)$$

that is $\vec{\nabla} \times \vec{A}^N = \vec{B}$ except along negative z-axis $(\theta = \pi)$



To remedy this problem, consider

$$A^S_x = \frac{gy}{r(r-z)}, \quad A^S_y = \frac{-gx}{r(r-z)}, \quad A^S_z = 0$$

$\longrightarrow \quad \vec{\nabla} \times \vec{A}^S = \vec{B}$ except along positive z-axis $(\theta = 0)$

remember: if $\vec{B} = \vec{\nabla} \times \vec{A}$ exactly, then

$$\Phi = \oint_S \vec{B} \cdot d\vec{S} = \oint_S \vec{\nabla} \times \vec{A} \cdot d\vec{S} = \int_V \underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})}_{=0} dV$$

In polar coordinates and differential form notation we get

$$\mathcal{A}_N = ig(1-\cos\theta)\,d\phi, \quad \mathcal{A}_S = -ig(1+\cos\theta)\,d\phi \qquad (*)$$

the transition function t_{NS} on $U_N \cap U_S$ is classified by $\pi(U(1)) = \mathbb{Z}$

$$\longrightarrow \quad t_{NS}(\phi) = \exp(i\varphi(\phi)), \quad \varphi : S^1 \to \mathbb{R}$$

then

$$\mathcal{A}_N = t_{NS}^{-1}\mathcal{A}_S t_{NS} + t_{NS}^{-1}\, d t_{NS}$$

$$= \mathcal{A}_S + i\,d\varphi$$

$$\longrightarrow \quad d\varphi = -i(\mathcal{A}_N - \mathcal{A}_S) \overset{(*)}{=} 2g\,d\phi$$

and so $\Delta\varphi = \int d\varphi = \int_0^{2\pi} 2g\,d\phi = 4\pi g$

$\rightarrow$ for t_{NS} to be well-deffined, need

$$\Delta\varphi/2\pi = 2g \in \mathbb{Z}$$

quantization condition for monopole charge

The total flux is

$$\Phi = \int_{S^2} \vec{B}\cdot d\vec{S} = \int_{U_N} dA_N + \int_{U_S} dA_S$$

$$= \int_{S^1} A_N - \int_{S^1} A_S = 2g\int_0^{2\pi} d\phi = 4\pi g$$

Charge quantization

Consider a Dirac fermion of charge e and mass m moving in the field of a mag. monopole of charge g

$$\gamma^\mu(\partial_\mu - ieA_\mu)\psi = m\psi$$

under the gauge trf. $A_\mu \mapsto A_\mu + \partial_\mu\Lambda$

ψ changes as

$$\psi \mapsto e^{ie\Lambda}\psi$$

As A^N and A^S differ by gauge trf.

$A^N_\mu - A^S_\mu = \partial_\mu(2g\phi)$, we get

$$\psi^S(\vec{r}) = \exp(-ie\Lambda)\psi^N(\vec{r})$$

taking $\theta = \frac{\pi}{2}$ and going around from $\phi = 0$ to $\phi = 2\pi$ gives

ψ single valued $\longrightarrow$ $2eg = n, \; n \in \mathbb{Z}$

"Dirac quantization cond."

$\longrightarrow$ if there exists a monopole somewhere in the universe, then $e = \frac{n}{2g}$ (5)

$\rightarrow$ all electric charges are quantized!

Electromagnetic duality

We saw that Maxwell's equations in vacuum are invariant under

$$(\vec{E} + i\vec{B}) \mapsto e^{i\theta}(\vec{E} + i\vec{B}) \qquad (*)$$

if magnetic charges exist, then Maxwell's eqs. are invariant under (*) even in the presence of charges!

(5) $\longrightarrow$ if e is small, g is large, and vice versa

eqs. for electric and magnetic charges are completely symmetric

$\longrightarrow$ for large e, perturbation theory breaks down $\longrightarrow$ consider instead the "magnetic dual" theory, where $g \ll 1$